

Multi-qubits and Polyvalent Singularity in Type II Supestring Theory

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Abstract

Inspired by geometric engineering method, we approach qubit systems in the context of D-branes in type II superstrings. Concretely, we establish a correspondence between such quantum systems and polyvalent singularities appearing in local Calabi-Yau manifolds. First, we examine 1-qubit by considering a D2-brane probing the $su(2)$ toric singularity associated with type IIA monovalent geometry. Then, we discuss the multi-qubits in terms of n factors of $su(2)$ singularities using the Cartan decomposition of non zero roots. Applying mirror symmetry, the 4-qubits are linked to the tetravalent singularity associated with the affine $\widehat{so}(8)$ Lie algebra matching with the ADE-correspondences in the context of quantum information theory. Precisely, these states can be identified with wrapped D4-branes in a Calabi-Yau 4-fold near such a singularity.

Keys words : String theory; Quantum information theory; Graph theory; Toric geometry; Calabi-Yau singularities; Lie algebras.

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1 Introduction

Quantum information theory (QIT) has attracted recently much attention mainly in relation with black holes and holography [1, 2, 3]. This theory, which has been considered as a possible combination of computer science and quantum mechanics, is based on a fundamental piece called qubit. This object has been dealt with by applying certain mathematical operations associated with tensor-product of the Hilbert vector spaces[4, 5, 6].

Qubit systems have been extensively studied using different approaches including superstring models and graph theory [8-16]. More precisely, a nice link between the stringy black holes and qubit systems have been investigated by considering the compactification scenario. More concretely, the supersymmetric STU black hole derived from the II superstrings has been linked to 3-qubits using the hyperdeterminant concept [8, 9]. This link, which is known by black hole/qubit correspondence, has been enriched by many extensions and generalizations to superqubits using supermanifold calculations[13]. In particular, a study based on toric complex geometry has been done leading to a classification of qubit systems in terms of black holes in type II superstrings using D-branes [16].

Alternative investigations have conducted using graph theory including Andinka which has been explored in the study of the supersymmetric representation theory. These graphs have been used to classify a class of qubits in terms of extremal black branes in type II superstrings [13]. Moreover, colored toric graphs of a product of \mathbb{CP}^1 projective spaces have been also used to deal with concepts of QIT including logic gates [15].

The present work aims to contribute to these activities by establishing a correspondence between multi-qubits and Calabi-Yau polyvalent singularities inspired by geometric engineering method. Concretely, these qubit systems are interpreted in terms D-branes of type II superstrings probing such singularities. First, we examine 1-qubit by considering a D2-brane probing $su(2)$ toric singularity associated with type IIA monovalent geometry. Then, the Cartan decomposition of non zero roots has allowed us to discuss the multi-qubits using n factors of $su(2)$ toric singularities. Applying mirror symmetry, the 4-qubits are linked with the tetravalent singularity associated with the affine $\widehat{so(8)}$ Lie algebra matching with the ADE-correspondences in the context of QIT. Precisely, these states can be identified with wrapped D4-branes in a Calabi-Yau 4-fold near such a singularity.

The organization of this paper is as follows. In section 2, we give a concise presentation of polyvalent singularities explored in the geometric engineering method. Section 3 concerns lower dimensional qubits in terms of D2-branes living in type IIA superstring. In section 4, a correspondence between multi-qubits and polyvalent singularities appearing in local Calabi-Yau manifolds is established by combining D-branes, graph theory, and Lie algebras. Section 5 is devoted to discussions, open questions and speculations supported by

Calabi-Yau singularities.

2 Polyvalent geometry in type IIA superstring

A nice framework to discuss polyvalent geometry is toric description of complex manifolds explored in type II superstring compactifications [17]. It is recalled that a n -dimensional toric manifold \mathbf{M}^n can be represented by a toric polytope $\Delta(\mathbf{V}^n)$ spanned by $k = n + r$ vertices v_i belonging to the n - dimensional lattice \mathbf{Z}^n [18, 19, 20]. These vertices satisfy r relations

$$\sum_{i=1}^{n+r} q_i^a v_i = 0, \quad a = 1, \dots, r, \quad (2.1)$$

where q_i^a are integers called Mori vectors carrying many physical and mathematical data. The building model is the one dimensional projective \mathbb{CP}^1 which can be represented by two vertices v_1 and v_2 on the real line. These two vertices verify the following condition

$$v_1 + v_2 = 0, \quad (2.2)$$

which corresponds to the north and the south poles of \mathbb{CP}^1 , respectively. In this case, the toric realization is just the segment $[v_1, v_2]$ linking the two vertices v_1 and v_2 . For higher dimensional geometries, the toric descriptions are slightly more complicated and can be found in [18, 19, 20]. It has been understood that toric geometry can be considered as a good place to study mirror symmetry discussed in the compactification of type II superstrings on Calabi-Yau manifolds [21, 22, 23]. In such compactifications, the mirror symmetry has been explored to study the deformation of the ADE singularities based on bivalent and trivalent geometries appeared in the toric realization of Calabi-Yau manifolds used in the geometric engineering method of gauge theories [17]. Following [21, 22, 23], the mirror geometry can be obtained using such a symmetry in the associated sigma model language. In this way, the mirror geometry is typically given by a superpotential in the dual Landau Ginzburg (LG) model

$$\begin{aligned} W &= P(x_1, \dots, x_{n-1}) = \sum_i y_i, \\ W &= uv \end{aligned} \quad (2.3)$$

subject to

$$\prod_{i=1}^{r+n} y_i^{q_i^a} = 1, \quad a = 1 \dots, r. \quad (2.4)$$

These equations have been handled to discuss the polyvalent singularities considered as an extension of the bivalent and trivalent geometries used in the geometric engineering method of supersymmetric theories in terms of D-branes probing toric singularities [17]. Before working out a general picture of such a geometry, it is useful to give a concise representation of lower dimensional cases involving bivalent and trivalent ones.

The leading example is the monovalent geometry appearing in the A_2 Dynkin diagram, where each vertex is connected only to another one. One of them is considered as the central vertex represented by the following Mori vector

$$q_i^a = (-2, 1, 0, \dots, 0). \quad (2.5)$$

The next example is the bivalent model known as linear geometry. This bivalent vertex appears in the mirror geometry of type IIA superstring on the A_n ($n \geq 3$) space family. In the geometric construction of supersymmetric quantum field theories, this vertex allows one to build a linear chain of gauge groups $\prod SU$ with bi-fundamental matter fields. In particular, it has been used to recover results of the Seiberg-Witten model from the type IIA superstring moving on a K3 surface fibered over on a chain of \mathbb{CP}^1 curves according to the A_n Dynkin graphs[17]. In toric geometry language, the corresponding Mori vectors take the following form

$$q_i^a = (-2, 1, 1, 0, \dots, 0). \quad (2.6)$$

In local geometries of the K3 surface known by the deformed ALE spaces, the bivalent vertices represent a linear chain of divisors with self intersection (-2) and intersect two adjacent divisors once with contribution $(+1)$.

The trivalent geometry, however, involves both bivalent and trivalent vertices which has appeared in different occasions. In string theory for instance, this geometry has been used to incorporate fundamental matters in the geometric construction of a linear chain of $\prod SU$ gauge groups [17]. Precisely, it helps to examine the affine ADE singularities explored in the study of the elliptic fibration of the local Calabi-Yau threefolds which produce $N = 2$ superconformal theories in four dimensions. In toric realization of the ALE spaces, the trivalent geometry contains a central divisor with self intersection (-2) intersecting three other divisors once with contribution $(+1)$. The corresponding Mori vector reads as

$$q_i = (-2, 1, 1, 1, 0, \dots, 0). \quad (2.7)$$

The Calabi-Yau condition requires that this Mori vector should be modified and takes the following form

$$q_i = (-2, 1, 1, 1, -1, 0, \dots, 0). \quad (2.8)$$

In type IIB superstring theory, the mirror geometry can be obtained by solving the constraint given in the equation (2.4). Indeed, one finds the following monomials

$$1, x_1, x_2, x_3, x_1 x_2 x_3. \quad (2.9)$$

In this solution, 1 corresponds to central divisor, while x_1 , x_2 and x_3 are associated with the remaining ones having contributions (+1). Here the terms $x_1 x_2 x_3$ corresponds to an auxiliary divisor with contribution (-1) introduced to cancel the first class of Chern required by the Calabi-Yau condition. A close inspection shows that the trivalent geometry is relevant in the study of the complex deformation of the T_{p_1, p_2, p_3} trivalent singularity defined as the intersection of three chains type A_{p_1-1} , A_{p_2-1} and A_{p_3-1} appearing in the blowing up of elliptic exceptional singularities $E_{6,7,8}$ [17]. They are given by the following elliptic curves, respectively,

$$\begin{aligned} T_{3,3,3} &: x_1^3 + x_2^3 + x_3^3 + \lambda x_1 x_2 x_3 \\ T_{2,4,4} &: x_1^2 + x_2^4 + x_3^4 + \lambda x_1 x_2 x_3 \\ T_{2,3,6} &: x_1^3 + x_2^2 + x_3^6 + \lambda x_1 x_2 x_3 \end{aligned} \quad (2.10)$$

where λ is a complex parameter.

In the fourthcoming sections, we establish a link between qubits and polyvalent type IIA geometry using D-branes wrapping on the associated spheres.

3 Lower dimensional qubits and polyvalent geometries in type IIA superstring theory

To start, it is recalled that the physics of qubit has been extensively investigated from different physical and mathematical aspects[4, 5, 6, 7]. Using Dirac notation, 1-qubit is described by the following state

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle \quad (3.1)$$

where a_i are complex numbers verifying the probability condition

$$|a_0|^2 + |a_1|^2 = 1. \quad (3.2)$$

It should be denoted that this condition can be interpreted geometrically in terms of the so-called Bloch sphere, \mathbb{CP}^1 . Similarly, the 2-qubits are represented by the state

$$|\psi\rangle = a_{00}|00\rangle + a_{10}|10\rangle + a_{01}|01\rangle + a_{11}|11\rangle \quad (3.3)$$

with the probability condition

$$|a_{00}|^2 + |a_{10}|^2 + |a_{01}|^2 + |a_{11}|^2 = 1. \quad (3.4)$$

This equation, however, defines a 3-dimensional complex projective space \mathbb{CP}^3 generalizing the Bloch sphere. This analysis can be extended to n -qubits associated with 2^n configuration states. Using the binary notation, the general state reads as

$$|\psi\rangle = \sum_{i_1 \dots i_n=0,1} a_{i_1 \dots i_n} |i_1 \dots i_n\rangle, \quad (3.5)$$

where $a_{i_1 \dots i_n}$ verify the real normalization condition

$$\sum_{i_1 \dots i_n=0,1} a_{i_1 \dots i_n} \bar{a}_{i_1 \dots i_n} = 1 \quad (3.6)$$

defining the \mathbb{CP}^{2^n-1} complex projective space. An inspection shows that the qubit systems can be represented by polyvalent geometry using type II D-branes. In what follows, we refer to it as n -valent geometry. More precisely, we will show that this is associated with the states describing the n -qubit systems. In this way, a quantum state is interpreted in terms of D2-branes wrapping spheres in type IIA superstring compactifications. We expect that the n -valent geometry should encode certain data on the corresponding brane physics offering a new take on the graphic representation of QIT using techniques based on a combination of toric geometry and graph theory. To understand how such a link could be true, we first examine the case of 1-qubit. Then, we give a general statement for the n -valent geometry. For the 1-qubit, the geometry is a local description of the K3 surface where the manifold develops a $su(3)$ singularity. It corresponds to vanishing two intersecting 2-spheres. Near such a singular point, the K3 surface can be viewed as an asymptotically locally Euclidean (ALE) complex space which is algebraically given by the blowing down curves of the A_2 singularity. This singularity can be removed by two intersecting \mathbb{CP}^1 curves according to the topology of the A_2 Dynkin graph. This picture is considered as a 1-valent geometry since each \mathbb{CP}^1 intersects only another one which can be represented by two vertices in the dual type IIB superstring theory. Deleting such a vertex, we get one vertex corresponding to the A_1 Dynkin graph associated with the toric $su(2)$ singularity of the K3 surface. The local geometry of this background is described by the algebraic complex equation

$$xy = z^2$$

where x, y, z are complex coordinates of type IIA geometry. The singularity can be deformed

by blowing up the singular point by a \mathbb{CP}^1 complex curve. In this way, a D2-brane wrapping around such a complex curve gives two states $|\pm\rangle$ depending on the two possible orientations for the wrapping procedure. In Lie algebras, this mapping can be supported by the root system decomposition of $su(2)$ symmetry given by

$$su(2) = h \oplus g_{+\alpha} \oplus g_{-\alpha} \quad (3.7)$$

where h is the associated Cartan subalgebra [24, 25, 26]. Concretely, these two states $|\pm\rangle$ correspond to the two dimensional vector space associated with the two roots $\{\pm\alpha\}$

$$E_\alpha = \frac{su(2)}{h} = g_{+\alpha} \oplus g_{-\alpha}. \quad (3.8)$$

Now, we consider the following corresponding

$$\begin{aligned} g_{+\alpha} : |+\rangle &\leftrightarrow |0\rangle \\ g_{-\alpha} : |-\rangle &\leftrightarrow |1\rangle \end{aligned} \quad (3.9)$$

which gives a mapping between qubit states and the states of the D2-brane wrapping configuration space. In this case, the probability of measuring the qubit in certain state could be determined in terms of winding numbers on \mathbb{CP}^1 . Moreover, it should be noted that the Weyl group of $su(2)$ Lie algebra which is $Z_2 = \{e, \sigma\}$ symmetry can be associated with the Pauli X gate, acting as a NOT gate. In two dimensional representation of Z_2 , one has

$$\sigma = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.10)$$

Having discussed 1-qubit case, we move now to the next model associated with the 2-valent geometry dealing with the 2-qubit systems defined in a 4 dimensional Hilbert space.

The 2-qubit model, which is interesting from entanglement applications, will be associated with the 2-valent geometry appearing in the A_3 type IIA superstring. It contains a central 2-sphere \mathbb{CP}^1 which intersects two other ones according to the A_3 Dynkin diagram in type IIB mirror description. Removing the central vertex, we get a graph which can be identified with the Dynkin graph of the $su(2) \oplus su(2)$ Lie algebra.

In this way, the corresponding type IIA geometry involves two isolated \mathbb{CP}^1 's. Two D2-brane wrapping around such a geometry gives four states $|\pm\rangle \otimes |\pm\rangle$ depending on the two possible orientations on each \mathbb{CP}^1 . In Lie algebra formwork, these configurations correspond to the four non zero roots $\{\pm\alpha^i, i = 1, 2\}$ of the $su(2) \oplus su(2)$ Lie algebra. This can be

supported by the root system decomposition providing a four dimensional vector space

$$E_\alpha = E_\alpha^1 \oplus E_\alpha^2 \quad (3.11)$$

where the factor E_α^i is given by

$$E_\alpha^i = g_{+\alpha^i} \oplus g_{-\alpha^i}, \quad i = 1, 2. \quad (3.12)$$

In this case, the Weyl group will be identified with the $Z_2 \times Z_2$ symmetry associated with the existence of four states in type IIA superstring using D2-branes wrapping on two isolated \mathbb{CP}^1 's.

4 Type IIA polyvalent geometry of multi-qubits

An inspection shows that the n -valent geometry can have also a nice graph theory realization by combining toric geometry and Lie algebra structures. Indeed, a graph G is represented by a pair of sets $G = (V(G), E(G))$, where $V(G)$ is the vertex set and $E(G)$ is associated with the edge set [27, 28, 29]. It is recalled that two vertices are adjacent if they are connected by a link. For any graph G , we define a symmetric squared matrix called an adjacency matrix $I(G) = (I_{ij})$, whose elements are either 0 or 1

$$I_{ij} = \begin{cases} 1, & (i, j) \in E(G), \\ 0, & (i, j) \notin E(G). \end{cases} \quad (4.1)$$

This matrix, which plays a primordial role to provide connections with many areas in mathematical and physics, encodes all the information residing on the graph. These data can be used to give a geometric representation for complicated systems including standard like models of particle physics, or more generally quiver gauge models built from string theory. A special example of graphs being relevant in the present work is called star graph formed by a central vertex that is connected to other outer vertices. In this way, the n -valent geometry can be represented by a star graph containing a central vertex connected with n ones. In Lie algebra theory, these graphs could represent indefinite Lie algebras generalizing the finite and affine symmetries. It is observed that for $n = 3$, we recover the trivalent geometry appearing in the finite $so(8)$ Lie algebra, known also by D_4 . For $n = 4$, however, we get the tetravalent vertex of the affine $\widehat{so(8)}$ Dynkin graph, known also by \widehat{D}_4 . More generally, the graph which represents the n -valent geometry could be worked out to extend the T_{p_1, p_2, p_3} trivalent singularity to T_{p_1, p_3, \dots, p_n} polyvalent singularities by considering a non trivial intersection of n chains of bivalent geometries formed by spheres in the Calabi-Yau manifolds.

These geometries go beyond the bivalent and trivalent ones explored to engineer $N = 2$ field models in four dimensions from type II superstring compactifications [17].

Roughly, we would like to extend the results presented in the previous section to the n -valent geometry. Concretely, we can do something similar for $n > 2$. In type IIA superstring, associated with the middle-degree cohomology, a n -valent geometry can be described by a central sphere C_0 , with self intersection (-2) intersecting n other spheres $(C_i, i = 1, \dots, n)$ with contribution $(+1)$. The corresponding intersection numbers read as

$$\begin{aligned} C_0.C_0 &= -2 \\ C_0.C_\ell &= 1 \quad \ell = 1, \dots, n. \end{aligned} \quad (4.2)$$

In relation with the adjacency matrix in graph theory and toric geometry, the intersection numbers of the n -valent geometry can be written as follows

$$\begin{aligned} C_i.C_j &= q_i^j \\ &= -2\delta_{ij} + I_{ij}. \end{aligned} \quad (4.3)$$

However, the Calabi-Yau condition requires that one should add an extra non compact cycle C_{n+1} with contribution $2 - n$. In string theory compactification on local Calabi-Yau manifolds, this cycle does not affect the corresponding physics. In this way, the Mori vector representing the central sphere should read as

$$q_i^0 = (-2, \underbrace{1, \dots, 1}_n, 2 - n). \quad (4.4)$$

As in graph theory approach, if we delete the central vertex, we get the so-called empty graph having n vertices. An inspection shows that this graph corresponds to the Dynkin diagram of a particular Lie algebra defined by n pieces of $su(2)$ denoted by $\underbrace{su(2) \oplus \dots \oplus su(2)}_n$. In this way, there are 2^n ways for n type II D-branes to wrap n distinguishable blowing up spheres in the type II superstrings. It is remarked that the $su(2)$ singularities are distinguishable giving rise 2^n possible inequivalent states $|\pm\rangle \otimes \dots \otimes |\pm\rangle$. The associated root system decomposition is

$$E_\alpha = E_\alpha^1 \oplus \dots \oplus E_\alpha^n \quad (4.5)$$

where $E_\alpha^i = g_{+\alpha^i} \oplus g_{-\alpha^i}$. Besides the root decomposition, the proposed link can be supported by the corresponding homotopy group given by

$$\Pi_2(\underbrace{A_1 \oplus \dots \oplus A_1}_n) = \underbrace{Z_2 \times \dots \times Z_2}_n. \quad (4.6)$$

This link may offer a novel way to study quantum geometry associated with quantum mechanical theory of strings and D-branes wrapping non trivial cycles in the Calabi-Yau manifolds.

5 Discussions and open questions

In this work, we have approached the multi-qubits in the context of type II superstrings. In particular, we have interpreted the multi-qubits in terms of polyvalent geometry appearing in the Calabi-Yau manifolds used in the geometric engineering method. First, we have examined the 1-qubit in terms of monovalent geometry in type IIA superstring. The geometry is no thing but the A_2 ALE space. Deleting one node associated with the central sphere, we have obtained just one 2-sphere in which a D2-brane can wrap to give two states depending on the two possible orientations of the 2-sphere. This scenario reproduces the states of an 1-qubit system. This has been supported by the Cartan decomposition of $\mathfrak{su}(2)$ Lie algebra producing a two dimensional vector space associated with the non zero roots. Then, we have given a general picture describing n -qubits in terms of polyvalent geometry using methods based on star graph operations and the Cartan decomposition of $\underbrace{\mathfrak{su}(2) \oplus \dots \oplus \mathfrak{su}(2)}_n$ Lie algebra. In this way, the n -qubits correspond to 2^n non zero roots of such a Lie algebra.

It is worth noting that the $n = 4$ could match with the work dealing with the ADE correspondence in the context of QIF [30, 31, 32]. In that work the 4-qubits are linked to the D_4 singularity. In our case, however, these systems are associated with the $\widehat{\mathfrak{so}(8)}$ affine Lie algebra. It has been remarked that this singularity can appear in the mirror of a toric Calabi-Yau manifold where the Mori vectors, up some details, are given in terms of the $\widehat{\mathfrak{so}(8)}$ Cartan matrix. Solving the toric data associated with the Dynkin graph of $\widehat{\mathfrak{so}(8)}$ Lie algebra, we find the following the mirror geometry

$$P(x_1, x_2, x_3, x_4, v) = x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_1 x_2 x_3 x_4 + w(x_1^2 + x_2^2 + x_3^2 + x_4^2) + w^2. \quad (5.1)$$

describing a quasihomogenous hypersurface in the weighted projective space $\mathbb{WCP}_{1,1,1,1,2}^4$. This manifold can be associated with the tetravalent singularity

$$T_{4,4,4,4} = x_1^4 + x_2^4 + x_3^4 + x_4^4 + \lambda x_1 x_2 x_3 x_4 \quad (5.2)$$

corresponding to a quartic in \mathbb{CP}^3 considered as a particular fibre identified with a K3 surface. This could produce a singular Calabi-Yau 4-fold with a K3-fibration developing a tetravalent singularity. Type IIA superstring on such a four-fold singularity can provide a similar statement. In this case, the compact part of the local deformed geometry is, topo-

logically, homotopic to a bouquet of four-spheres according to the $\widehat{so(8)}$ Dynkin graph. The 4-qubit states can be identified with wrapped D4-branes in such a Calabi-Yau 4-fold near the tetravalent singularity.

In the general case, we expect that the n -valent geometry ($n > 2$) could appear in the $n - 2$ dimensional Calabi-Yau manifold fibrations of $(2n - 4)$ -folds. The fiber Calabi-Yau manifolds can be considered as homogeneous hypersurfaces in $n - 1$ -dimensional projective spaces. The n -qubit states can be determined in terms of wrapped $(2n - 4)$ -branes in a Calabi-Yau $(2n - 4)$ -folds near a polyvalent singularity.

This work comes up with many open directions and speculations. The intersecting problem is the discussion of other quantum concepts including the entanglement using polyvalent singularities of type II superstring compactifications. We believe that the central vertex may play a primordial role in the study of such a concept. Using graph theory, it should be interesting to compute the entanglement entropies of some edges with respect to the rest of the star graph which be useful to better understand the physical characteristics and mathematical structures of entangled states from Calabi-Yau polyvalent singularities. It has been observed that there could be possible directions for future investigation in connection with quadrality for supersymmetric models based on mirror symmetry and toric geometry realization of local Calabi-Yau manifolds discussed more recently in [33]. This will be addressed elsewhere.

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